1. Volume and Work

A container is created by revolving the curve $y = x^2$ from y=0 to y= 9 about the y-axis.
a. Write an integral that computes V(h), the volume of liquid contained if the container is filled to a height h.
b. How much water does this container hold when it is full?
c. To what height does the water level reach when the volume is half-full?

d. If the container is full of water, how much work does it take to pump all of the water out of the container? Use the symbols ρ and g in your computations to represent density of water and acceleration of gravity. If you'd like a number at the end, you can estimate $\rho = 1000kg/m^3$ and $g = 10m/s^2$

e. Your pump breaks down after pumping out half of the volume of water in the tank. What proportion of the work required to pump all the water out was done? *(Hint: use your answer from above, and think about pumping the water out from the top of the tank down to the remaining water level.)*

2. Integration by Parts

Let p(x) be an abstract function defined on the interval [a, b] with the following properties:

- p(a) = p(b) = 0
- p''(x) exists for each x in [a,b].

a. Use these properties to show that

$$\int_a^b p(x)p''(x)\,dx = -\int_a^b (p'(x))^2\,dx$$

. *(Hint, integrate the left side by parts and simplify).*

As another side note, it will be useful to use the integration by parts formula if the integrals have limits:

$$\int_a^b f(x)g'(x)\,dx + \int_a^b f'(x)g(x)\,dx = f(x)g(x)\Big|_a^b$$

. Note that the right side of the equation is evaluated from a to b.

b. In addition to the properties above, suppose p'' is proportional to p. In symbols. p''(x) = Kp(x), for some constant K. Show that K must be a negative number.

c. We will apply these results to work on a common integral next week. For now, we just think about functions that satisfy each of the following criteria:

- Make up two functions, along with an interval [a, b] for each so that f(a) = f(b) = 0.
- Make up two different functions so that f''(x) = K f(x) for some constant K.

For each of the four functions you wrote down, test each condition.