Numerical Methods-Assignment-1
Mathematical Prelimanaries, Taylor's Theorem, Order of Convergence

(1) Let L be a real number and let $\{a_n\}$ be a sequence of real numbers. If there exists a positive integer N such that

$$|a_n - L| \leq \lambda |a_{n-1} - L|$$

for all $n \geq N$ and for some fixed $\lambda \in (0,1)$, then show that $a_n \to L$ as $n \to \infty$.

(2) Show that the equation $\sin x + x^2 = 1$ has at least one root in the interval $[0,1]$.

(3) Let $f : [0,1] \to [0,1]$ be a continuous function. Prove that the equation $f(x) = x$ has at least one root lying in the interval $[0,1]$.

(4) Let f be continuous on $[a,b]$, let $x_1, \ldots, x_n$ be points in $[a,b]$, and let $g_1, \ldots, g_n$ be real numbers having same sign. Show that, for some $\xi \in [a,b]$,

$$\sum_{i=1}^{n} f(x_i) g_i = f(\xi) \sum_{i=1}^{n} g_i.$$

(5) Let g be a continuously differentiable function such that the equation $g(x) = 0$ has at least n distinct real roots. Show that the equation $g'(x) = 0$ has at least $(n-1)$ distinct real roots.

(6) Let $f : [a,b] \to \mathbb{R}$ be given by $f(x) = x^2$. Find a point c specified by the mean-value theorem for derivatives. Verify that this point lies in the interval $[a,b]$.

(7) In the second mean-value theorem for integrals, let $f(x) = e^x$, $g(x) = x$, $x \in [0,1]$. Find a point c specified by the theorem and verify that this point lies in the interval $(0,1)$.

(8) (i) Use Taylor's formula about $a = 0$ to evaluate approximately the value of the function $f(x) = e^x$ at $x = 0.5$ using three terms (i.e., $n = 2$) in the formula.
(ii) Obtain the remainder $R_2(0.5)$ in terms of the unknown c and and also obtain the remainder estimate when $x \in (0, 0.5)$.

(9) Obtain Taylor's expansion for the function $f(x) = \sin x$ about the point $a = 0$ when $n = 1$ and $n = 5$. Find the reminder term in both the cases and obtain their remainder estimates when $x \in [0,1]$.

(10) Prove or disprove:
(i) $2n^2 + 3n + 4 = o(n)$ as $n \to \infty$.
(ii) $\frac{n+1}{n^2} = o(\frac{1}{n})$ as $n \to \infty$.
(iii) $\frac{n+1}{n^3} = O(\frac{1}{n})$ as $n \to \infty$.
(iv) $\frac{e^n}{n^5} = O(\frac{1}{n})$ as $n \to \infty$.
(v) $e^x - 1 = O(x^2)$ as $x \to 0$.
(vi) $x^{-2} = O(\cot x)$ as $x \to 0$.

(11) Assume that $f(h) = p(h) + O(h^n)$ and $g(h) = q(h) + O(h^m)$, for some positive integers n and m, and as $h \to 0$. Find a positive integer r such that

$$f(h) + g(h) = p(h) + q(h) + O(h^r).$$