Numerical Methods-Problem Sheet-5
Matrix theory and Iterative methods

(1) Show that the norm defined on the set of all $n \times n$-matrices by

$$\|A\| := \max_{\substack{1\le i\le n\\ 1\le j\le n}} |a_{ij}|$$

is not subordinate to any vector norm on $\mathbb{R}^n$.

(2) Let A be an invertible matrix. Show that its condition number $\kappa(A)$ satisfies $\kappa(A) \geq 1$.

(3) Let A be an $n \times n$ matrix with real entries. Let $\kappa_2(A)$ and $\kappa_\infty(A)$ denote the condition numbers of a matrix A that are computed using the matrix norms $\|A\|_2$ and $\|A\|_\infty$, respectively. Answer the following questions.
(i) Determine all the diagonal matrices such that $\kappa_\infty(A) = 1$.
(ii) Let Q be a matrix such that $Q^T Q = I$ (such matrices are called orthogonal matrices). Show that $\kappa_2(Q) = 1$.

(4) In solving the system of equations $A\mathbf{x} = \mathbf{b}$ with matrix

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 2.01 \end{pmatrix}$$

estimate the relative error in the solution vector $\mathbf{x}$ in terms of the relative error in $\mathbf{b}$. Test your estimate in the case when $\mathbf{b} = (4,4)^T$ and $\tilde{\mathbf{b}} = (3,5)^T$. Use the maximum norm for vectors in $\mathbb{R}^2$.

(5) Write the formula for the Gauss-Jacobi iterative sequence of the system

$$\begin{aligned} 7x_1 - 15x_2 - 21x_3 &= 2, \\ 7x_1 - x_2 - 5x_3 &= -3, \\ 7x_1 + 5x_2 + x_3 &= 1. \end{aligned}$$

Without performing the iterations, show that the sequence does not converge to the exact solution of this system. Can you make a suitable interchange of rows so that the resulting system is diagonally dominants?

(6) Find the $n \times n$ matrix B and the $n-$dimensional vector $\mathbf{c}$ such that the Gauss-Seidal method can be written in the form

$$\mathbf{x}^{(k+1)} = B\mathbf{x}^{(k)} + \mathbf{c},\ k = 0,1,2,...,$$

for a given system $A\mathbf{x} = \mathbf{b}$, where A is an $n \times n$-matrix and $\mathbf{b} \in \mathbb{R}^n$.

(7) Let $\mathbf{x}^{(7)}$ be the 7^{th} term of the Gauss-Seidel iterative sequence for the system

$$\begin{aligned} 3x_1 + 2x_2 &= 1, \\ 4x_1 + 12x_2 + 3x_3 &= -2, \\ x_1 + 3x_2 - 5x_3 &= 3, \end{aligned}$$

with $\mathbf{x}^{(0)} = (0,0,0)^T$. If $\mathbf{x}$ denotes the exact solution of the given system, then show that

$$\left\|\mathbf{e}^{(7)}\right\|_\infty \leq 0.058527664\, \|\mathbf{x}\|_\infty .$$