First Homework

Curves and surfaces, MATH2040

January 25, 2021

This is the first homework sheet for the course curves and surfaces, it is due February 5^{th}.

I would like to remind you that any solutions you hand in for this (and future) homework sheet(s) must be written by you, and may not be copied from anyone else. You are encouraged to discuss these problems with classmates, the TA, and the instructor, but with no one else. Please refer to the course outline, the honesty declaration, and the university guidelines for more information.

Problem 1. **(a)** State the inverse function theorem for functions of a single variable. (You don't need to prove it.)

(b) Let $I \subseteq \mathbb{R}$ be an open interval, and let $g : I \to \mathbb{R}$ be a smooth function. Let J be the range of g, i.e. $J = g(I)$. Prove that if $g'(t) \neq 0$ for all $t \in I$, then there exists a <u>smooth</u> function $f : J \to I$ which is the inverse to g, i.e. $g \circ f = \mathbb{1}_J$, and $f \circ g = \mathbb{1}_I$. Hint: Use part (a).

(c) Prove that the function f has the property that, for all $t \in J$, we have

$$f'(t) = \frac{1}{g'(f(t))}. \tag{1}$$

Problem 2. Let $\alpha : I \to \mathbb{R}^3$ be a regular smooth curve, (not necessarily parametrized by arc length). Let $g : I \to \mathbb{R}$ be the arc length of α. Set $J = g(I)$, and let f be the inverse of g. For the existence of g, we refer to Problem 1. Recall that the curve $\beta = \alpha \circ f : J \to \mathbb{R}^3$ is parametrized by arc length, (you don't need to prove this). By definition, the curvature of α at $s \in I$, is the curvature of β at f^{-1}, i.e. $k_\alpha(s) = k_\beta(f^{-1}(s))$. In this problem, we're going to find a formula for the curvature of α.

(a) Prove that, for all $s \in I$ we have

$$\beta''(f^{-1}(s)) = \alpha''(s) \cdot (f'(f^{-1}(s)))^2 + \alpha'(s) \cdot f''(f^{-1}(s)).$$

(b) Use Eq. 1 to express $f'(g(s))$ in terms of $\|\alpha'(s)\|$, and then use the result to prove that, for all $s \in I$

$$f''(g(s)) = -\frac{\alpha''(s) \cdot \alpha'(s)}{\|\alpha'(s)\|^4}.$$

HINT: First prove that

$$\frac{\mathrm{d}}{\mathrm{d}s}\|\alpha'(s)\| = \frac{\alpha''(s) \cdot \alpha'(s)}{\|\alpha'(s)\|}.$$

(c) Use parts (a) and (b) to show that

$$\|\beta''(f^{-1}(s))\|^2 = \frac{1}{\|\alpha'(s)\|^6}\left(\|\alpha''(s)\|^2\|\alpha'(s)\|^2 - (\alpha''(s) \cdot \alpha'(s))^2\right),$$

and, then take the square root of both sides of this equation, and use an appropriate formula for the cross product, to prove

$$k_\alpha(s) = \frac{\|\alpha''(s) \times \alpha'(s)\|}{\|\alpha'(s)\|^3}. \tag{2}$$

Problem 3. Consider the smooth curve $\alpha : \mathbb{R} \to \mathbb{R}^2$, defined as

$$\alpha(t) = \frac{\lambda e^{\lambda t}}{\sqrt{\lambda^2 + 1}}(\cos(t), \sin(t)).$$

(a) Determine its velocity $\alpha'(t)$, and show that α is not parametrized by arc length.

(b) Determine the curvature $k_\alpha(t)$ of α. HINT: use Equation 2.

(c) Find a reparametrization of α so that it is parametrized by arc length. That is, find a smooth bijective function $f : I \to \mathbb{R}$ with smooth inverse $f^{-1} : \mathbb{R} \to I$, such that the smooth curve $\beta = \alpha \circ f : I \to \mathbb{R}^2$ is parametrized by arc length. (HINT: Start by determining the arc length of α.)

(d) Determine the curvature of β, and compare with part (b).

Problem 4. Consider the smooth curve $\alpha : (0, \infty) \to \mathbb{R}^3$, defined as

$$\alpha(t) = (e^{-\frac{t}{8\pi}} \cos(t), e^{-\frac{t}{8\pi}} \sin(t), t).$$

(a) Sketch the trace of α, and in the same figure, draw the tangent line of α at $t = \pi/2$.

(b) Compute the curvature k_α and the torsion τ_α of α.

(c) Set $a(t) = \alpha''(t)$, and define

$$a^\perp(t) = a(t) - \frac{a(t) \cdot \alpha'(t)}{\|\alpha'(t)\|}.$$

Recall that the unit normal is then defined by

$$n_\alpha(t) = \frac{a^\perp(t)}{\|a^\perp(t)\|}.$$

Calculate $n_\alpha(t)$.

(d) The *osculating circle* of α at $t_0 \in (0, \infty)$ is the trace of the curve

$$\gamma(s) = \frac{1}{k_\alpha(t_0)} \left(\cos(s) \frac{\alpha'(t_0)}{\|\alpha'(t_0)\|} + \sin(s) n_\alpha(t_0) \right) + \alpha(t_0) + \frac{1}{k_\alpha(t_0)} n_\alpha(t_0).$$

Set $t_0 = \pi$ and draw the corresponding osculating circle Draw the trace of α in the same picture.