Probability and Statistics - AUC - ASSIGNMENT III

(*Due March 15, 14:00. The 4 problems have equal weight.*)

For the problems below, do not only give your final answer but also show how you obtained it. In particular, indicate which rules and properties you have used.

For common distributions (Normal, Poisson, Geometric, etc.), derivation of expectations and variances is not required, you can use ready-made formulas.

Exercise III.1. Let X and Y be independent random variables, $X \sim \text{Uniform}(0, 1)$ and $Y \sim \text{Exponential}(1)$. Find the pdf of $Z = Y/X$. Can you calculate Z-related probabilities as areas of regions of the plane, like e.g. in exercise 3.52 from Rice? Why (not)?

Exercise III.2. A supermarket gives its customers a dino picture for every €5 spent. There are 10 different pictures and each time a customer receives a picture it is equally likely to be one of the 10 types. Denote by S the amount of money a customer will spend until pictures of all 10 types are present in their collection. Determine $\mathbb{E}S$ and $\mathbb{V}S$. Proceed as follows.

Suppose that the customer's collection already includes i types of pictures, $i = 1, 2, \ldots, 9$. Denote by N_i the number of pictures the customer will collect until they get a new type of picture (one of the remaining $10 - i$ types).

(a) Give (and motivate) the pmf of N_i. Which common distribution is this? (Specify the name and the parameter(s) value(s).)

(b) Write S as a suitable function of the N_i's and, based on this representation, determine $\mathbb{E}S$ and $\mathbb{V}S$.

Exercise III.3. The random vector (X, Y) has the joint pdf

$$f(x,y) = \begin{cases} x\mathrm{e}^{-x}\mathrm{e}^{-xy}, & \text{if } x, y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the conditional pdf of Y given $X = x$. Which common distribution is this? (Specify the name and the parameter(s) value(s).)

(b) Determine $\mathbb{E}(Y|\,X)$ and $\mathbb{V}(Y|\,X)$.

(c) Determine $\mathbb{P}(2 < Y < 3|X = 1)$.

Exercise III.4. Suppose that a unit of capital has been invested, and that the interest is compounded yearly. That is, in $n = 1, 2, \ldots$ years, the invested capital will grow to $\prod_{i=1}^{n}(1 + R_i/100)$ units, where R_i is the interest rate (in percentage) applied in year i of the investment. The future interest rates R_i are modelled as mutually independent random variables with the normal $N(3, 1)$ distribution.

To answer questions (a) and (b), use the following properties. If $X_1, \ldots, X_n$ are mutually independent random variables, then

$$\mathbb{E}\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} \mathbb{E}X_i.$$

Also, if $X_1, \ldots, X_n$ are mutually independent random variables, then any transforms $g_1(X_1), \ldots, g_n(X_n)$ are mutually independent, too.

(a) Compute the expected capital size n years after the investment.

(b) Suppose a client has invested a unit of capital and is going to leave the money in the account for $N \sim$ Poisson(10) years. Compute the expected capital size at the time of withdrawal of this client.
Hint: Use the identity $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$.

(c) Compute the variance of the capital size (exactly) 10 years after the investment. Is it true that $\mathbb{V}(\prod_{i=1}^{n} X_i) = \prod_{i=1}^{n} \mathbb{V}X_i$ for independent random variables X_i?