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Problem Set 1

Problem (*An extension of Example 1 of Lecture 1*)

Suppose the economy in Example 1, Lecture 1 lasts for three quarters. Similar to Example 5 of Lecture 1, consider a security that pays $d_t = \$1$ if the economy state in quarter t is G and $d_t = \$0$ if the economy state in quarter t is B.

1. What is the sample space Ω?
   Answer: $\Omega = \{GGG, GGB, GBG, GBB, BGG, BGB, BBG, BBB\}$.

2. Following Example 1, find the filtration that corresponds to the σ-algebras $\mathcal{F}_t$ at $t = 0, 1, 2, 3$.
   Answer: $\mathcal{F}_0 = \{\emptyset, \Omega\}, \mathcal{F}_1 = \{\emptyset, \Omega, \{GGG, GGB, GBG, GBB\}, \{BGG, BGB, BBG, BBB\}\}$,$\mathcal{F}_2 = \{\emptyset, \Omega, \{GGG, GGB\}, \{GBG, GBB\}, \{BGG, BGB\}, \{BBG, BBB\}$, all possible unions of these sets$\}$, $\mathcal{F}_3 = 2^\Omega$, where 2^Ω is the set of all subsets of Ω.

3. Calculate the probability measure $\mathbb{P}$ that is associated with each $\sigma-$algebra $\mathcal{F}_t$ above for $t = 0, 1, 2, 3$.
   Answer:

   (a) For $\mathcal{F}_0$: $\mathbb{P}(\emptyset) = 0, \mathbb{P}(\Omega) = 1$.

   (b) For $\mathcal{F}_1$: $\mathbb{P}(\emptyset) = 0, \mathbb{P}(\Omega) = 1$,

   $$\mathbb{P}(\{GGG, GGB, GBG, GBB\}) = q, \mathbb{P}(\{BGG, BGB, BBG, BBB\}\}) = 1 - q$$

   (c) For $\mathcal{F}_2$: $\mathbb{P}(\emptyset) = 0, \mathbb{P}(\Omega) = 1$,

   $$\mathbb{P}(\{GGG, GGB\}) = q^2, \mathbb{P}(\{GBG, GBB\}) = q(1 - q),$$

   $$\mathbb{P}(\{BGG, BGB\}\}) = q(1 - q), \mathbb{P}(\{BBG, BBB\}\}) = (1 - q)^2$$

   $$\mathbb{P}(\{GGG, GGB, GBG, GBB\}) = q^2 + q(1 - q) = q$$

   $$\mathbb{P}(\{GGG, GGB, BGG, BGB\}) = q^2 + q(1 - q) = q$$

   $$\mathbb{P}(\{GGG, GGB, BBG, BBB\}) = q^2 + (1 - q)^2 = 1 - 2q + 2q^2$$

   $$\mathbb{P}(\{GBG, GBB, BGG, BGB\}) = 2q(1 - q)$$

   $$\mathbb{P}(\{GBG, GBB, BBG, BBB\}) = q(1 - q) + (1 - q)^2 = 1 - q$$

   $$\mathbb{P}(\{BGG, BGB, BBG, BBB\}) = q(1 - q) + (1 - q)^2 = 1 - q$$

   (d) For $\mathcal{F}_3$: Similar calculations as above.

4. Consider a security X with date-3 payoff defined as

$$X = d_1 + d_2 + d_3$$

Let Y be a put option on X with a strike price of $K = 2$ and maturity of $T = 3$. Recall that the payoff for this put option is $Y = \max(K - X, 0)$.

(a) Describe Y as a map: $Y : \Omega \to \mathbb{R}$.
Answer:

$$Y(GGG) = Y(GGB) = Y(GBG) = Y(BGG) = 0$$
$$Y(BBG) = Y(BGB) = Y(GBB) = 1$$

and

$$Y(BBB) = 2$$

(b) Find the smallest $\sigma-$algebra that makes Y a random variable.
Answer: It's the $\sigma-$algebra generated by Y:

$\{\emptyset, \{BBB\}, \{GBB, BGB, BBG\}, \{GGG, GGB, GBG, BGG\}, \{BBB, GBB, BGB, BBG\},$
$\{BBB, GGG, GGB, GBG, BGG\}, \{GBB, BGB, BBG, GGG, GGB, GBG, BGG\}, \Omega\}$