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Stochastic Foundation of Finance (FIN 538)
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Problem Set 4

Problem 1 In this problem, you will use the following steps to show that $Var(\int_0^t C_s dB_s) = \int_0^t \mathbb{E}[C_s^2]ds$ where C_t is adapted to the natural filtration associated with the Brownian motion B_t:

1. Consider the finite sum $S_n = \sum_{i=1}^{n} C_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})$ so that $\lim_{n\to\infty} S_n = \int_0^t C_s dB_s$. Show that
$$Var(S_n) = \sum_{i=1}^{n} Var(C_{t_{i-1}}(B_{t_i} - B_{t_{i-1}}))$$
2. Write
$$Var(C_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})) = \mathbb{E}[(C_{t_{i-1}}(B_{t_i} - B_{t_{i-1}}))^2] - (\mathbb{E}[C_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})])^2$$
Use the law of iterated expectation to show that $\mathbb{E}[C_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})] = 0$ and $\mathbb{E}[(C_{t_{i-1}}(B_{t_i} - B_{t_{i-1}}))^2] = \mathbb{E}[C_{t_{i-1}}^2](t_i - t_{i-1})$.
3. Take the limit when $n \to \infty$ to conclude that $Var(\int_0^t C_s dB_s) = \int_0^t \mathbb{E}[C_s^2]ds$.

Problem 2 1. Show that $Var(\int_0^1 B_s^2 dB_s) = 1$.

2. We will now verify that $Var(\int_0^1 B_s^2 dB_s) = 1$ using simulations. Use Excel to generate 100 sample paths of the Brownian motion in the time interval $[0,1]$ by following the following steps:
   (a) Partition the interval $[0,1]$ into 100 small intervals each of length 0.01: $t_0 = 0, t_1 = 0.01, t_2 = 0.01, \cdots, t_{100} = 1$.
   (b) For each t_i, generate a random number $B_{t_i} = B_{t_{i-1}} + \sqrt{t_i - t_{i-1}} * N(0,1)$ where $N(0,1)$ is the standard normal random variable.
   (c) Compute the sum $S(\omega) = \sum_{i=1}^{100} B_{t_{i-1}}^2(\omega)(B_{t_i}(\omega) - B_{t_{i-1}}(\omega))$
   (d) Repeat the above process 100 times to get 100 sample paths of the Brownian motion. Compute
   $$\bar{S}_1 = \frac{\sum_\omega S(\omega)}{100}, \bar{S}_2 = \frac{\sum_\omega S^2(\omega)}{100},$$
   (e) Use the approximation $Var(\int_0^1 B_s^2 dB_s) \approx \mathbb{E}[S^2] - \mathbb{E}[S]^2 \approx \bar{S}_2 - (\bar{S}_1)^2$. Is your answer close to 1?

Please submit your code in the .xlsx together with your answers.