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Stochastic Foundation of Finance (FIN 538)
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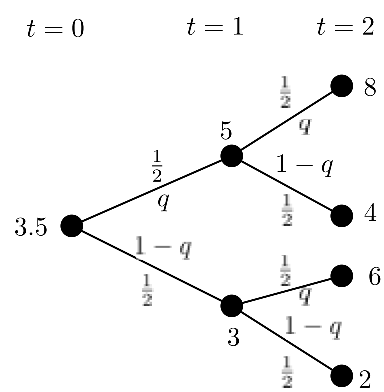
Problem Set 5

Problem 1 Consider a world with only two dates: today and tomorrow. There are two possible states tomorrow: Good and Bad. There are two different risky stocks A, B *and no other assets* in the market. Assume there is no arbitrage. The probability of the two states and the current prices and future (state-contingent) prices of the assets are listed below.

Asset	Current Price	Tomorrow Price	
		Good (Prob. 30%)	Bad (Prob. 70%)
A	\$3	\$5	\$2
B	\$2	\$4	\$1

1. Construct a portfolio of A and B in which the risk associated with A is exactly offset by the risk of B (that is, use B to hedge A). In other words, this portfolio is risk free.
2. Compute the risk-free rate in the market.
3. Does the risk-free rate depend on the prospect of the economy (that is, the probabilities of Good and Bad states)? Explain why or why not.

Problem 2 Consider the price of a security that follows the process in the figure below. At each time $t = 0, t = 1$, the price jumps up or down with a (physical) probability $\frac{1}{2}$. The risk-free rate is 0 in both periods.



1. Are investors who trade this security risk-averse, risk-neutral or risk-loving?
2. Find the probability q (of the price going up) at which investors can price the security as if they're risk neutral.

Problem 3 Use Ito's Lemma to compute the following differentials:

1. $d\left(t^2 B_t^{\frac{1}{2}}\right)$.
2. $d\left(e^{\int_0^t \mu ds + \sigma dB_s}\right)$ where μ, σ are constants.

Problem 4

1. Let the process X_t satisfy $dX_t = \mu dt + \sigma dB_t$. Show that X_t is a martingale if and only if $\mu = 0$.
2. Using Ito's lemma, find the expression for dY_t where $Y_t = e^{\theta B_t - \frac{1}{2}\theta^2 t}$, θ is a constant and show that Y_t is a martingale.
3. Using Ito's lemma, find the expression for dZ_t where $Z_t = (B_t - t)e^{B_t - \frac{t}{2}}$ and show that Z_t is a martingale.

Problem 5 Consider the following process

$$dr_t = (\theta - ar_t)dt + \sigma dB_t$$

where θ, a, and σ are constant, and B_t is a standard Brownian motion. Define the process $R_t = e^{at} r_t$.

1. Express dR_t in differential form using Ito's lemma. This expression for dR_t should depend only on t and B_t (and not r_t or R_t). Use this result to express R_t in integral form (the expression for R_t might depend on a stochastic integral and it's fine to leave it like that.)
2. Solve for r_t (similarly, the expression for r_t might depend on a stochastic integral and it's fine to leave it in that form.)