Linear Approximation and Differentials

(1) True or False. Explain why or why not.
Also, for each of (i)-(iv), graph f and L (if it exists) on one set of axes.
(i) The linear approximation to $f(x) = x^2$ at $x = 0$ is $L(x) = 0$.
(ii) Linear approximation at $x = 0$ provides a good approximation to $f(x) = |x|$.
(iii) If $f(x) = mx + b$, then the linear approximation to f at any point is $L(x) = f(x)$.
(iv) When linear approximation is used to estimate the value of $f(x) = \ln x$ near $x = e$ the approximations are underestimates of the true value.

(2) Use linear approximation to estimate $f(5.1)$ given that $f(5) = 10$, and $f'(5) = -2$.

(3) Given a function $f(x) = (1 + x)^n$, show that $L(x) = 1 + nx$ is the linear approximation of f at 0.

(4) Consider the function $f(x) = \sqrt{2}\cos x$.
(i) Find the linear approximation L to the function f at $a = \frac{\pi}{4}$.
(ii) Graph f and L on the same set of axes.
(iii) Based on the graphs of part (ii), state whether linear approximations to f near a are underestimates or overestimates.
(iv) Compute $f''(a)$ to confirm your conclusion.

(5) Use linear approximations to estimate the following quantities.
Choose a suitable function f and a value of a that produces a small error.
(i) $\sqrt[3]{-7.97}$
(ii) $e^{0.02}$

(6) Differentials. Consider the function $f(x) = \ln(1 - x)$.
(i) Express the relationship between a small change in x and the corresponding change in y in the form $dy = f'(x)dx$.
(ii) Use your answer in part (i) to approximate the change in f when x changes from $x = -1$ to $x = -1.02$.