1. For each of the statements below determine whether it is true or false, providing reasons for your answer.

(a) For the real line $\mathbb{R}$ with the standard topology and its subset $\mathbb{Q}$ of all rational numbers we have $\mathrm{Cl}(\mathbb{Q}) = \mathbb{R}$. [2 marks]

(b) For the real line $\mathbb{R}$ with the standard topology and its subset $\mathbb{Q}$ of all rational numbers we have $\partial(\mathbb{Q}) = \mathbb{R}$. [2 marks]

(c) Singleton sets always closed in Hausdorff spaces. [2 marks]

2. Let

$$\mathcal{T} = \{(a, \infty) : a \in [-\infty, \infty]\}.$$

(Note: when $a = -\infty$ we have $(a, \infty) = \mathbb{R}$, while if $a = \infty$, then $(a, \infty) = \emptyset$.)

(a) Show that $\mathcal{T}$ is a topology on $\mathbb{R}$. [5 marks]

(b) Carefully explain whether $\mathcal{T}$ is Hausdorff or not. [3 marks]

3. Let X be a topological space and let $K_1, K_2, \ldots, K_m$ be compact subsets of X. Show that

$$K = K_1 \cup K_2 \cup \ldots \cup K_m$$

is compact, too. [5 marks]

4. Let X be a topological space. Prove that

$$\mathrm{Int}(A \cap B) = \mathrm{Int}(A) \cap \mathrm{Int}(B)$$

for all subsets A and B of X. [8 marks]

5. Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be two topological spaces and $f, g : X \to Y$ be two continuous maps. Show that, if $(Y, \mathcal{S})$ is Hausdorff, the set

$$\Upsilon = \{x \in X : f(x) \neq g(x)\}$$

is open. [5 marks]

6. Let $(X, \mathcal{T})$ be a Hausdorff topological space and let

$$K_1 \supseteq K_2 \supseteq \ldots \supseteq K_n \supseteq \ldots$$

be an infinite sequence of **non-empty compact** subsets of X. Show that

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

i.e. there exists a point $x \in X$ such that $x \in K_n$ for all $n \geq 1$. [8 marks]