1. For each of the statements below determine whether it is true or false, providing reasons for your answer. Let $(X, \mathcal{T})$ be a topological space and $A, B \subseteq X$ be two compact subsets.

(a) Is $A \cup B$ compact ? [2 marks]

(b) If $(X, \mathcal{T})$ is Hausdorff, is $A \cap B$ compact ? [2 marks]

2. Let $f : [a, b] \to \mathbb{R}$ be continuous.

(a) Prove that the map $g : [a, b] \to \mathbb{R}^2$, $x \mapsto (x, f(x))$ is also continuous. [3 marks]

(b) Prove that the graph of f is a compact subset of $\mathbb{R}^2$. [3 marks]

3. Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be connected topological spaces. Show that $X \times Y$ is connected in the product topology. [8 marks]

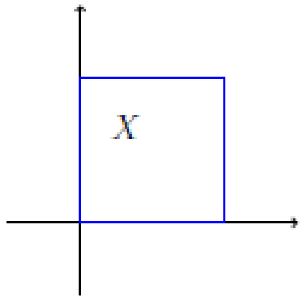
4. Consider the subsets [5 marks]

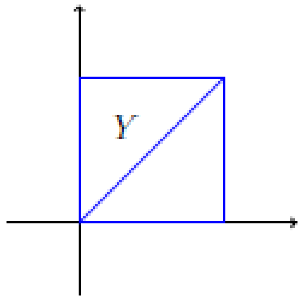
$$X = \{(x, y) : 0 \leq x \leq 1\,,\ y \in \{0, 1\}\} \cup \{(x, y) : 0 \leq y \leq 1\,,\ x \in \{0, 1\}\}$$

and

$$Y = X \cup \{(x, x) : 0 \leq x \leq 1\}$$

of $\mathbb{R}^2$ with the standard topology from $\mathbb{R}^2$. Show that X and Y are not homeomorphic.





5. Consider the subset [8 marks]

$$Z = \{(0, 0), (0, 1)\} \cup \bigcup_{n=1}^{\infty} L_n$$

of $\mathbb{R}^2$ (with the standard topology from $\mathbb{R}^2$), where

$$L_n = \{(1/n, y) : 0 \leq y \leq 1\}$$

for all $n \geq 1$. Let U be a non-empty subset of Z which is both open and closed in Z. Show that if U contains one of the points $(0, 0)$ and $(0, 1)$, then it contains the other as well.

6. Let $\mathbb{RP}^2$ be the real projective space as defined in lectures, and let $\pi : S^2 \longrightarrow \mathbb{RP}^2$ be the natural projection. Consider $\mathbb{R}^3$ and $\mathbb{R}^4$ with the standard topologies, and let the function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ be defined by

$$f(x, y, z) = (x^2 - z^2, xy, yz, xz).$$

(a) Show that there exists a function $g : \mathbb{RP}^2 \longrightarrow \mathbb{R}^4$ such that $f = g \circ \pi$ on S^2. [4 marks]

(b) Show that g is continuous and one-to-one. [11 marks]

(c) Show that g is a homeomorphism between $\mathbb{RP}^2$ and the subset $Y = g(\mathbb{RP}^2)$ of $\mathbb{R}^4$. [4 marks]