Introduction to Matrices and Algebraic Systems

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PREFACE

Matrices and algebraic systems have a wide area of application in engineering and exact sciences. There is a great variety of books about this theme. However, many times the person interested in using these techniques for application has a lot of difficulty in finding what he needs. This happen, mainly due to the fact that the great majority of books available gives more emphasis to the inherent logical structure formalism than to the physical interpretation of the said theories.

This notebook had its origin in lecture notes used to complement two undergraduate courses given by the author. The choice of the themes chosen here were guided by the students' necessity in obtaining familiarity with those topics of matrices and algebraic systems that were used more frequently in the practical solution of exact sciences problems without much worry about the formalism inherent in its logical structure. It is also presented here, as far as possible applications of the exposed themes the solution of practical problems.

Much care was taken in order to present the subject here in the most simple and intuitive manner as possible. In some cases there is an excess of detail that may even look redundant. The intention of such an approach was to avoid the omission of algebraic details and make the assimilation of the exposed topics here explained as easy as possible. It is expected that all the material exposed in this book can be easily assimilated by senior high school students and above.

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Linear systems appear in all mathematical models of physical processes and sometimes indirectly in the numerical solution of other mathematical models. These applications occur in practically all areas of sciences: Physical, biological, social, etc. More over linear systems are directly involved in the areas of optimization, approximation of functions of nonlinear systems, numerical solutions of differential equations and innumerous other types of problems. The presentation and algebraic manipulation of linear systems becomes more concise and compact if made through the use of matrices.

A matrix **A** of order m by n can be represented as a set of m by n numbers disposed in lines and columns as shown below:

$$\tilde{A} = A_{mn} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{1.1}$$

In the notation above m is the number of lines and n is the number of columns of the matrix. The matrix above can also be expressed in indicial form ion the following way:

$$\tilde{A} = a_{ij} \quad ; \qquad 1 \le i \le m \quad ; \qquad 1 \le j \le n \tag{1.2}$$

1.1 – Matrix algebra

It is possible to make algebraic operations with matrices, in the same way that it is done with scalars; however there are pre-defined rules for each type of algebraic operation done with matrices.

Addition and subtraction of matrices: Let A_{mn} be a matrix of order m by n and B_{jk} a matrix of order J by k. The addition and subtraction operation can only be made between matrices with the same number of lines and same number of columns. So, if m=j and n=k then it is then possible to define the operations of addition and subtraction between the matrices:

$$\tilde{Q} = q_{mn} = a_{mn} \pm b_{mn} = b_{mn} \pm a_{mn} \tag{1.1.1}$$

$$= \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & \cdots & a_{1n} \pm b_{1n} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & \cdots & a_{2n} \pm b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} \pm b_{m1} & a_{m2} \pm b_{m2} & \cdots & a_{mn} \pm b_{mn} \end{bmatrix}$$

The matrix **Q** is also a matrix of order m by n, given by:

$$q_{rs} = a_{rs} \pm b_{rs} = b_{rs} \pm a_{rs} \quad ; \quad 1 \le r \le m \quad ; \quad 1 \le s \le n \tag{1.1.2}$$

Example Consider the two matrices **A** and **B** given:

$$\tilde{A} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \quad ; \quad \tilde{B} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix} \tag{1.1.3}$$

Since they are of the same order it is then possible to make the operations of sum and subtraction between them:

$$\tilde{S} = s_{mn} = a_{mn} + b_{mn} = \begin{bmatrix} 1+1 & 2+4 \\ 4+2 & 5+5 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 6 & 10 \end{bmatrix} \qquad (1.1.4)$$

$$\tilde{D} = d_{mn} = a_{mn} - b_{mn} = \begin{bmatrix} 1-1 & 2-4 \\ 4-2 & 5-5 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \qquad (1.1.5)$$

Multiplication of a matrix by a scalar: Let **A** be a matrix of order m by n and λ a scalar. The product of λ by **A** is given by:

$$\tilde{P} = \lambda \tilde{A} = \lambda a_{mn} = \lambda \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{bmatrix} \qquad (1.1.6)$$

Example: Consider a matrix **A** of equation (1.1.3) and the scalar $\lambda = 2$. In this case we have:

$$\tilde{M} = 2\tilde{A} = 2a_{mn} = 2\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2*1 & 2*2 \\ 2*4 & 2*5 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 8 & 10 \end{bmatrix} \qquad (1.1.7)$$

Transposed matrix: Considere a matrix **A** of n lines and m columns. The transpose of **A** is given by is denoted by $\mathbf{A^T}$ is the matrix obtained from A changing its columns by its rows. So if

$$\tilde{A} = a_{ij} \quad \text{then} \quad \tilde{A}^T = a_{ji} \quad ; \quad 1 \le i \le n \quad ; \quad 1 \le j \le m \qquad (1.1.8)$$

Example: Consider the following matrix of order 3 by 2:

$$\tilde{A} = \begin{bmatrix} 3 & 2 \\ 2 & 5 \\ 4 & 6 \end{bmatrix} \qquad (1.1.9)$$

In this case the transposed matrix $\mathbf{A^T}$ is given by:

$$\tilde{A}^T = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 5 & 6 \end{bmatrix} \qquad (1.1.10)$$

Square matrix: A matrix **A** is said to square if the number of lines is equal to the number of columns, or:

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad ; \quad n = m \qquad (1.1.11)$$

Some properties of matrix operation: Let **A** and **B** be two matrices of order m by n and c and d scalars. Then:

$$c(\tilde{A} + \tilde{B}) = c\tilde{A} + c\tilde{B} \quad \text{and} \quad d(c\tilde{A}) = dc\tilde{A} \qquad (1.1.12)$$

And also:

$$c\tilde{A} = c\tilde{B} \quad \Rightarrow \quad \tilde{A} = \tilde{B} \qquad (1.1.13)$$

Multiplication of matrices: Let **A** and **B** be matrices of order n by m and r by s, respectively:

$$\tilde{A} = a_{mn} \quad \text{and} \quad \tilde{B} = b_{rs} \qquad (1.1.14)$$

If n = r or if the number of columns of **A** is equal to the number of rows of **B**, then it is possible to define the product **AB** in the following manner:

$$\tilde{P} = \tilde{A}\,\tilde{B} = p_{\lambda k} = \sum_{j=1}^{n} a_{\lambda j}\, b_{jk} \quad ; \quad 1 \le \lambda \le m \quad ; \quad 1 \le k \le s \qquad (1.1.15)$$

The order of matrix **P** is m by s.

Example: Consider the two matrices **A** and **B** given by:

$$\tilde{A} = \begin{bmatrix} 1 & 2 \end{bmatrix} \quad ; \quad \tilde{B} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix} \qquad (1.1.16)$$

The number of columns of **A** is equal to the number of rows of **B** and it is possible multiply **B** by **A** on the left:

$$\tilde{A}\,\tilde{B} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 * 1 + 2 * 2 & 1 * 4 + 2 * 5 \end{bmatrix} = \begin{bmatrix} 5 & 14 \end{bmatrix} \qquad (1.1.17)$$

Observe that in this case the multiplication **BA** cannot be done.

Order of factors: Let **A** and **B** be two square matrices. Then in this case both products **A B** and **B A** can be calculated. However, in the multiplication of matrices the order of factors is important (not indifferent). Generally **A B** ≠ **B A**. Consider two given matrices A and B:

$$\tilde{A} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix} \quad ; \quad \tilde{B} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \qquad (1.1.18)$$

The product **A B** is:

$$\tilde{A}\,\tilde{B} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1*1+4*4 & 1*2+4*5 \\ 2*1+5*4 & 2*2+5*5 \end{bmatrix} = \begin{bmatrix} 17 & 22 \\ 22 & 29 \end{bmatrix} \qquad (1.1.19)$$

The product **B A** is:

$$\tilde{B}\,\tilde{A} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1*1+2*2 & 1*4+2*5 \\ 4*1+5*2 & 4*4+5*5 \end{bmatrix} = \begin{bmatrix} 5 & 14 \\ 14 & 41 \end{bmatrix} \qquad (1.1.20)$$

As it can be seen in this case **A B** ≠ **B A**. If **A B = B A** then the two matrices are said to be **Commutative**.

Some properties of the product of matrices: Consider three matrices **A**, **B** and **C**:

- If it is possible to calculate **A** (**B C**) and (**A B**) **C** then **A** (**B C**) = (**A B**) **C**
- If it is possible to calculate **A C** and **B C** then (**A + B**) **C** = **A C** + **B C**.
- If it is possible to calculate **C A** and **C B** then **C** (**A** + **B**) = **C A** + **C B**.

1.2 – Elementary matrices

When one works with matrices and algebraic systems there are some types of matrices that come up with frequency. We shall present some of these now:

Identity matrix: Na identity matrix **I** of order n is a matrix that has zero in all of its elements except the diagonal elements which are equal to 1 (one), or:

$$\tilde{I} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \cdots & \\ & & & 1 \end{bmatrix} \qquad (1.2.1)$$

Or

$$\tilde{I} = I_{ij} = a_{ij}\,\delta_{ij} \qquad (1.2.2)$$

In the notation above δ_{ij} is the Kronecker delta:

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \qquad (1.2.3)$$

The product of a square matrix A of order n by an identity matrix of the same order does not alter its value. In other words the identity matrix is the neutral element in the product of matrices:

$\tilde{I}\,\tilde{A} = \tilde{A}\,\tilde{I} = \tilde{A}$ (1.2.4)

Null matrix: A null matrix **0** of order n is a matrix which has zero in all of its elements:

$$\tilde{0} = 0_{ij} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \qquad (1.2.5)$$

Or

$$\tilde{0} = O_{ij} = 0 \quad ; \quad 1 \le i \le n \quad ; \quad 1 \le i \le n \qquad (1.2.6)$$

The null matrix is neutral element in the addition of matrices of same order. Given a matrix **A** of order n and a null matrix **0** of the same order then we have:

$$\tilde{A} \pm \tilde{0} = \tilde{0} \pm \tilde{A} = \tilde{A} \qquad (1.2.7)$$

Symmetric matrix: Let A be a square matrix of order n. The same is said to be symmetric if it is equal to its transpose:

$$\tilde{A} = a_{ij} = A^T = a_{ji} \quad ; \quad 1 \le i \le n \quad ; \quad 1 \le j \le n$$

or

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \tilde{A}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \qquad (1.2.8)$$

Example: Consider the following square matrix of order 3:

$$\tilde{A} = \begin{bmatrix} 1 & 2 & 5 \\ p & 3 & 4 \\ q & r & 2 \end{bmatrix} \qquad (1.2.9)$$

In this case A is symmetric if and only if p = 2, q = 5 and r = a. So we have that:

$$\tilde{A} = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \\ 5 & 4 & 2 \end{bmatrix} \qquad (1.2.10)$$

It is easy to note that:

$$\tilde{A}^T = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \\ 5 & 4 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \\ 5 & 4 & 2 \end{bmatrix} = \tilde{A} \quad \Rightarrow \quad \tilde{A} = \tilde{A}^T \qquad (1.2.11)$$

Observe that a matrix can only be symmetric if it is a square matrix.

Inverse matrix: An inverse matrix $\mathbf{A}^{-1}$ is a square matrix of order n such that when multiplied by another square matrix **A** of order n produces the identity matrix:

$$\tilde{A}\,\tilde{A}^{-1} = \tilde{A}^{-1}\,\tilde{A} = \tilde{I} \qquad (1.2.12)$$

Orthogonal matrix: A matrix **A** is said to be orthogonal if its transpose $\mathbf{A}^{T}$ is equal to its inverse:

$$\tilde{A}^{T} = \tilde{A}^{-1} \quad \Rightarrow \quad \tilde{A}^{T}\,\tilde{A} = \tilde{A}\,\tilde{A}^{T} = \tilde{I} \qquad (1.2.13)$$

Example: Consider the following matrix **A** given:

$$\tilde{A} = \begin{bmatrix} \cos(\theta) & -\operatorname{sen}(\theta) \\ \operatorname{sen}(\theta) & \cos(\theta) \end{bmatrix} \qquad (1.2.14)$$

The transpose of matrix **A** is given by:

$$\tilde{A}^{T} = \begin{bmatrix} \cos(\theta) & \operatorname{sen}(\theta) \\ -\operatorname{sen}(\theta) & \cos(\theta) \end{bmatrix} \qquad (1.2.15)$$

Doing the multiplication A by $\mathbf{A}^{T}$ we have:

$$\tilde{A}\,\tilde{A}^T = \begin{bmatrix} \cos(\theta) & -\operatorname{sen}(\theta) \\ \operatorname{sen}(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & \operatorname{sen}(\theta) \\ -\operatorname{sen}(\theta) & \cos(\theta) \end{bmatrix} =$$

$$= \begin{bmatrix} \cos^2(\theta) + \operatorname{sen}^2(\theta) & \cos(\theta)\operatorname{sen}(\theta) - \operatorname{sen}(\theta)\cos(\theta) \\ \operatorname{sen}(\theta)\cos(\theta) - \cos(\theta)\operatorname{sen}(\theta) & \operatorname{sen}^2(\theta) + \cos^2(\theta) \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \tilde{I} \qquad (1.2.16)$$

As it can be seen the transpose of matrix **A** given by equation (1.2.14) is its inverse matrix. In this case the matrix **A** shown before is the rotation matrix. Every rotation matrix is orthogonal.

Diagonal matrix: A square matrix **D** of order n is said to be diagonal if all of its elements outside of the diagonal are null:

$$\tilde{D} = d_{ij} = \delta_{ij} \quad \text{para} \quad 1 \le i \le n \quad ; \quad 1 \le j \le n$$

or

$$\tilde{D} = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \cdots & \\ & & & a_{nn} \end{bmatrix} \qquad (1.2.17)$$

Upper diagonal matrix: A matrix is said to be upper diagonal if all of its elements bellow the diagonal elements are null

$\tilde{U} = u_{ij} = 0 \quad \text{se} \quad i > j$

Or

$$\tilde{U} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \cdots & \cdots \\ & & & a_{nn} \end{bmatrix} \qquad (1.2.18)$$

Lower diagonal matrix: A matrix **L** is said to be lower diagonal if all of its elements above the diagonal are zero:

$\tilde{L} = l_{ij} = 0 \quad \text{se} \quad i < j$

ou

$$\tilde{L} = \begin{bmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \cdots & \cdots & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad (1.2.19)$$

Tridiagonal matrix: A tridiagonal matrix **T** is a particular type of matrix which has non zero elements in the main diagonal and in the diagonal superior and inferior:

$$\tilde{T} = \begin{bmatrix} a_{11} & a_{12} & & & \\ a_{21} & a_{22} & a_{23} & & \\ & & \cdots & & \\ & & a_{n-1n-2} & a_{n-1n-1} & a_{n-1n} \\ & & & a_{nn-1} & a_{nn} \end{bmatrix} \qquad (1.2.20)$$

Vector: A vector is a matrix which has only one culomun or only one row:

$$\tilde{A} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \quad \text{ou} \quad \tilde{A} = \begin{bmatrix} a_1 & a_1 & \cdots & a_n \end{bmatrix}^T \qquad (1.2.21)$$

In this note the symbol used for vectors is a bold face with a little arrow above. So if $\tilde{A}$ then it can be written as $\vec{A}$.

Block matrix: A block matrix A is one constituted of block of matrices with the same number of lines.

Example: Consider the following matrices:

$$D = \begin{bmatrix} 1 & 2 & 1 \\ 5 & 2 & 4 \\ 1 & 3 & 3 \end{bmatrix} \quad ; \quad \tilde{A} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \tag{1.2.22}$$

The block matrix constructed from the two matrices above is:

$$\tilde{B} = \left[\tilde{D} \vdots \tilde{A} \right] = \begin{bmatrix} 1 & 2 & 1 & \vdots & 1 \\ 5 & 2 & 4 & \vdots & 3 \\ 1 & 3 & 3 & \vdots & 2 \end{bmatrix} \tag{1.2.23}$$

1.3 – Determinant of a matrix

Every square matrix **A** has a number which is a function of its elements, denominated determinant of A. The numerical value of this determinant can be calculated by Laplace expansion:

$$\det(\tilde{A}) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(\tilde{A}_{ij}) \quad \text{para } \forall\, i \in [1, n]$$

or (1.3.1)

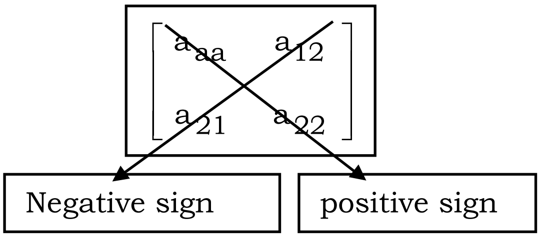
$$\det(\tilde{A}) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(\tilde{A}_{ij}) \quad \text{para } \forall\, j \in [1, n]$$

Where A_{ij} is the matrix obtained from A by taking off its i^{nd} line and j^{nd} row. Using expression 1.3.1 we get:

$$\det(\tilde{A}) = (-1)^{1+1} * a_{11} * a_{22} + (-1)^{2+1} * a_{12} * a_{21} \qquad (1.3.2)$$

$$= a_{11} * a_{22} - a_{12} * a_{21}$$

Observing the expressions above it can be noted that Laplace expansion for a matrix of order two can be calculated in following way:



(1.3.3)

Using Cramer's rule:

$$\det(\tilde{A}) = +(a_{11} * a_{22}) - (a_{12} * a_{21}) \qquad (1.3.4)$$

Example 1: Consider the following matrix of order two:

$$\tilde{A} = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} \qquad (1.3.5)$$

For a matrix of order two Laplace's expansion is:

$$\det(\tilde{A}) = (-1)^{1+1} * a_{11} * a_{22} + (-1)^{1+2} * a_{12} * a_{21}$$

(1.3.6)

$$= a_{11} * a_{22} - a_{12} * a_{21}$$

Using the rule given by equation (1.3.4), we have:

$$\det(\tilde{A}) = (-1)^{1+1} * 2 * 2 + (-1)^{1+2} * 2 * 1$$

(1.3.7)

$$= 2 * 2 - 2 * 1 = 1$$

Using expression (1.3.4), we have:

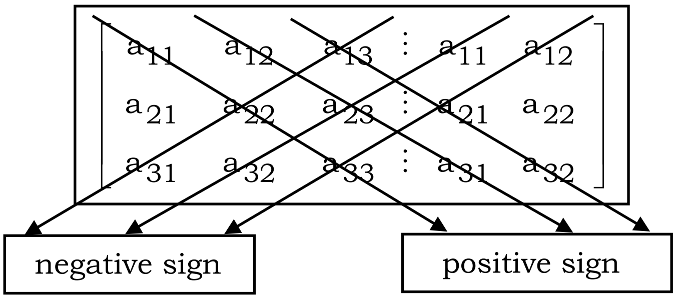
$$\det(\tilde{A}) = +(2 * 2) - (2 * 1) = (4) - (3) = 1 \quad (1.3.8)$$

As it can be seen the results are equal in both cases.

To calculate the determinant of a matrix of order three using Laplace's expansion we have:

$$\begin{aligned} \det(\tilde{A}) = {} & (-1)^{1+1} * a_{11} * (a_{22} * a_{33} - a_{23} * a_{32}) \\ & + (-1)^{1+2} * a_{12} * (a_{21} * a_{33} - a_{23} * a_{31}) \quad (1.3.9) \\ & + (-1)^{1+3} * a_{13} * (a_{21} * a_{32} - a_{22} * a_{31}) \end{aligned}$$

Analogous for the case of a square matrix of order two it is possible to stablish a practical rule for the calculation of the determinant of this matrix:



(1.3.10)

In this case we have:

$$\det(\tilde{A}) = +(a_{11} * a_{22} * a_{33} + a_{12} * a_{23} * a_{31} + a_{13} * a_{21} * a_{32})$$
$$- (a_{13} * a_{22} * a_{31} + a_{11} * a_{23} * a_{32} + a_{12} * a_{21} * a_{33}) \quad (1.3.11)$$

Example 2: Consider the following matrix of order three:

$$\tilde{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \qquad (1.3.12)$$

Using the expression of equation (1.3.9), we have:

$$\det(\tilde{A}) = (-1)^{1+1} * 1 * (3 * 2 - 1 * 1)$$
$$+ (-1)^{1+2} * 1 * (1 * 2 - 1 * 3) \qquad (1.3.13)$$
$$+ (-1)^{1+3} * 1 * (1 * 1 - 3 * 3)$$

$$\Rightarrow \quad \det(\tilde{A}) = (6 - 1) - (2 - 3) + (1 - 9) = (5) - (-1) + (-8) = -2 \qquad (1.3.14)$$

Using expression (1.3.11), we hve:

$$\det(\tilde{A}) = +(1 * 3 * 2 + 1 * 1 * 3 + 1 * 1 * 1) - (1 * 3 * 3 + 1 * 1 * 1 + 1 * 1 * 2)$$
$$= +(6 + 3 + 1) - (9 + 1 + 2) = +(10) - (12) = -2 \qquad (1.3.15)$$

As it can be observed the obtained by equation (1.3.11) is identical to the value obtained using Laplace's expansion.

Meaning of a matrix determinant: Consider a square matrix A. If the determinant is equal to zero then the matrix is singular and doesn't have an inverse. In other words if det (**A**) = 0 then there does not exist a matrix $\mathbf{A}^{-1}$ such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\, A^{-1} = \underline{I}$